Intro. to ODEs

Quiz 8 **Solutions**

1) Solve the following initial value problem.

$$\begin{aligned} 2\frac{d^2x}{dt^2} - 5\frac{dx}{dt} - 3x &= 0 \\ x(0) &= 0 \\ x'(0) &= 7 \end{aligned}$$

The characteristic polynomial is $2r^2 - 5r - 3 = (2r+1)(r-3) = 0$ which gives roots $r = -1/2, 3$. So, the general solution is

$$x(t) = c_1 e^{-t/2} + c_2 e^{3t}.$$

The initial conditions give the equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ -\frac{1}{2}c_1 + 3c_2 &= 7, \end{aligned}$$

which have solutions $c_1 = -2$ and $c_2 = 2$. So, the solution to the given IVP is

$$x(t) = -2e^{-t/2} + 2e^{3t}.$$

2) Solve the following initial value problem.

$$\begin{aligned} t^2\frac{d^2x}{dt^2} - 6x &= 0 \\ x(1) &= 0 \\ x'(1) &= 1 \end{aligned}$$

The characteristic polynomial is $r(r-1) - 6 = r^2 - r - 6 = (r+2)(r-3) = 0$ which gives roots $r = -2, 3$. So, the general solution is

$$x(t) = \frac{c_1}{t^2} + c_2 t^3.$$

The initial conditions give the equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ -2c_1 + 3c_2 &= 1, \end{aligned}$$

which have solutions $c_1 = -1/5$ and $c_2 = 1/5$. So, the solution to the given IVP is

$$x(t) = -\frac{1}{5t^2} + \frac{t^3}{5}.$$

3) Find the general solution to the following differential equation.

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 5e^{2t}$$

The characteristic polynomial for the homogeneous solution is $r^2 + 4r + 13 = (r+2)^2 + 9 = 0$ which gives roots $r = -2 \pm 3i$. So, the homogeneous solution is

$$x_H(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t).$$

For a particular solution, we try $x_P(t) = Ae^{2t}$. Running this through the differential equation gives

$$4Ae^{2t} + 4\left(2Ae^{2t}\right) + 13Ae^{2t} = 25Ae^{2t} = 5e^{2t}$$

which means $A = 1/5$. So, the general solutions is

$$x(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + \frac{1}{5}e^{2t}.$$

4) Find the general solution to the following differential equation.

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 16t^2$$

The characteristic polynomial for the homogeneous solution is $r^2 + 5r + 4 = (r+4)(r+1) = 0$ which gives roots $r = -4, -1$. So, the homogeneous solution is

$$x_H(t) = c_1 e^{-4t} + c_2 e^{-t}.$$

For a particular solution, we try $x_P(t) = At^2 + Bt + C$. Running this through the differential equation gives

$$2A + 5\left(2At + B\right) + 4\left(At^2 + Bt + C\right) = 4At^2 + (10A + 4B)t + (2A + 5B + 4C) = 16t^2.$$

This means we need to solve the following equations.

$$\begin{aligned} 4A &= 16 \\ 10A + 4B &= 0 \\ 2A + 5B + 4C &= 0 \end{aligned}$$

The solutions are easily seen to be $A = 4, B = -10$, and $C = 21/2$. So, the general solutions is

$$x(t) = c_1 e^{-4t} + c_2 e^{-t} + 4t^2 - 10t + \frac{21}{2}.$$